Setting Limits in the Presence of Nuisance Parameters

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The Problem

- x events in the signal region
- y events in data sidebands (or from MC), measured with some uncertainty, statistical and systematic
- z a measurement of the efficiency,

  measured with some uncertainty, statistical and systematic

→ How do we set limits on the signal rate?

Previous Solution : Cousins-Highland

- Basically, integrate out the nuisance parameter.
- Problem 1: “hidden” Bayesian method – what should be used as a prior?
- Problem 2: does it work, i.e. does it have coverage? Recent studies (Conrad and Tegenfeldt) suggest some overcoverage.

New Solution –
Profile Likelihood

Need some notation:

- x – number of events in signal region
- y - number of events in data sidebands
- τ – relative "size" of background region to signal region, so that y/ τ is estimate of background in signal region

- m – number of MC events to test efficiency
- z – number of MC events that survive the cuts

So z/m is an estimate of the efficiency

Unknown Parameters:

- μ – signal rate (what we want to know)
- b – background rate in signal region
- e – efficiency

Probability Model:

$X \sim Pois(e\mu+b)$, $Y \sim Pois(\tau b)$, $Z \sim Binom(m,e)$

Loglikelihood:

$l(\mu,b,e) = (-2)*\ (x\log(e\mu+b)-\log(x!)-(e\mu+b) +$

$y\log(\tau b)-\log(y!)-\tau b +$

$\log(m!)-\log(z!)-\log((m-z)!)+z\log(e)+(m-z)\log(1-e))$

is a function of all parameters

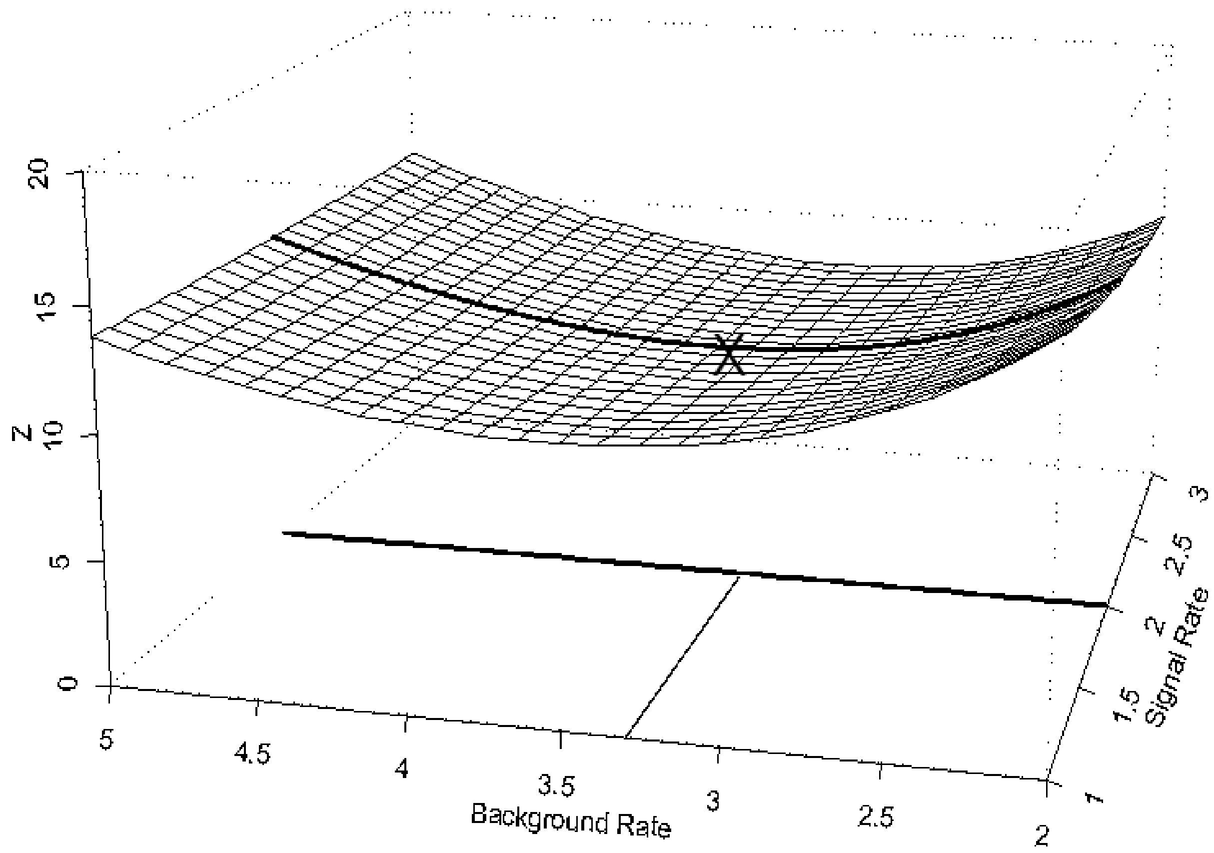
Idea: for each μ find b and e which make the observations most likely – profile likelihood

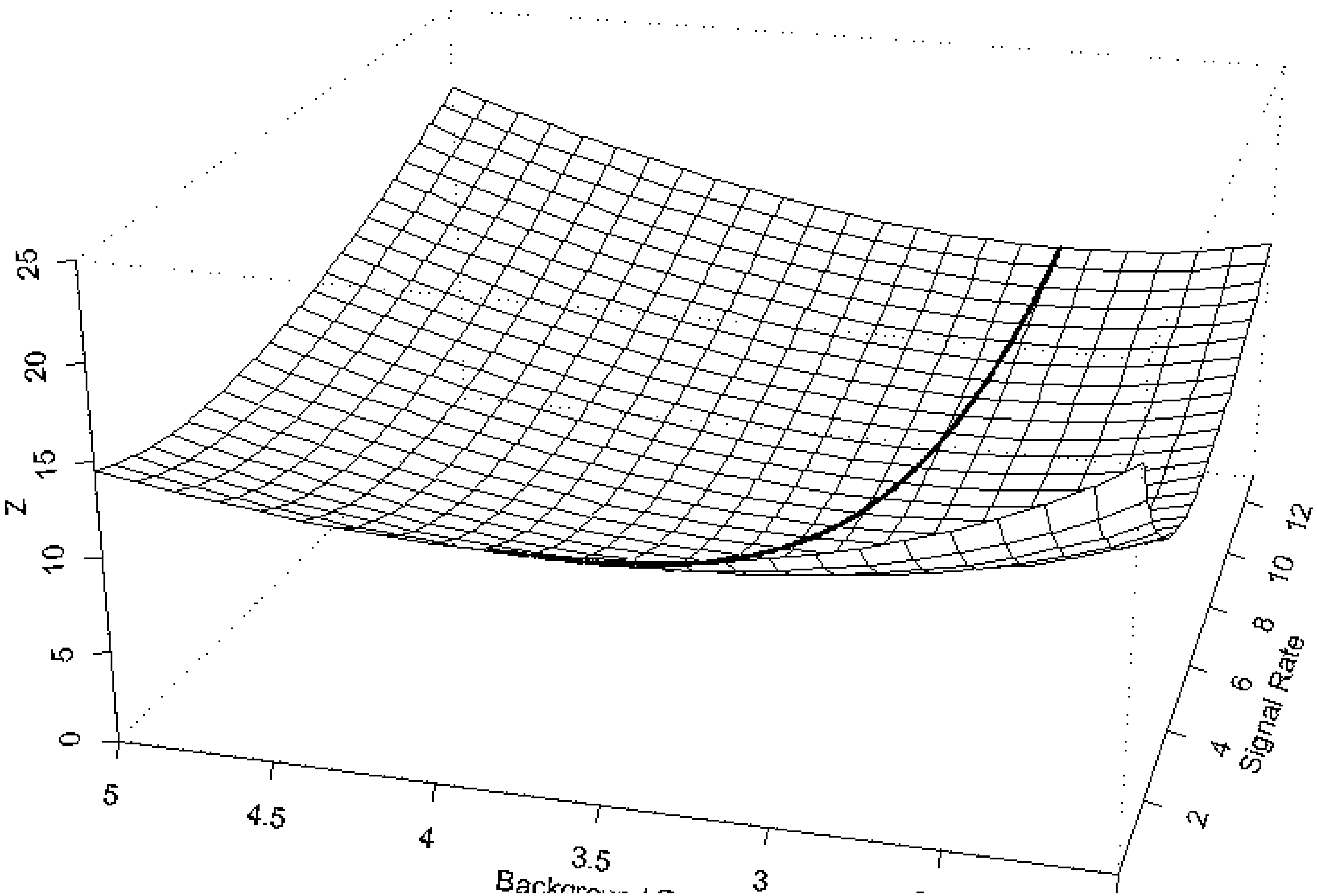
Illustration of Profile Likelihood

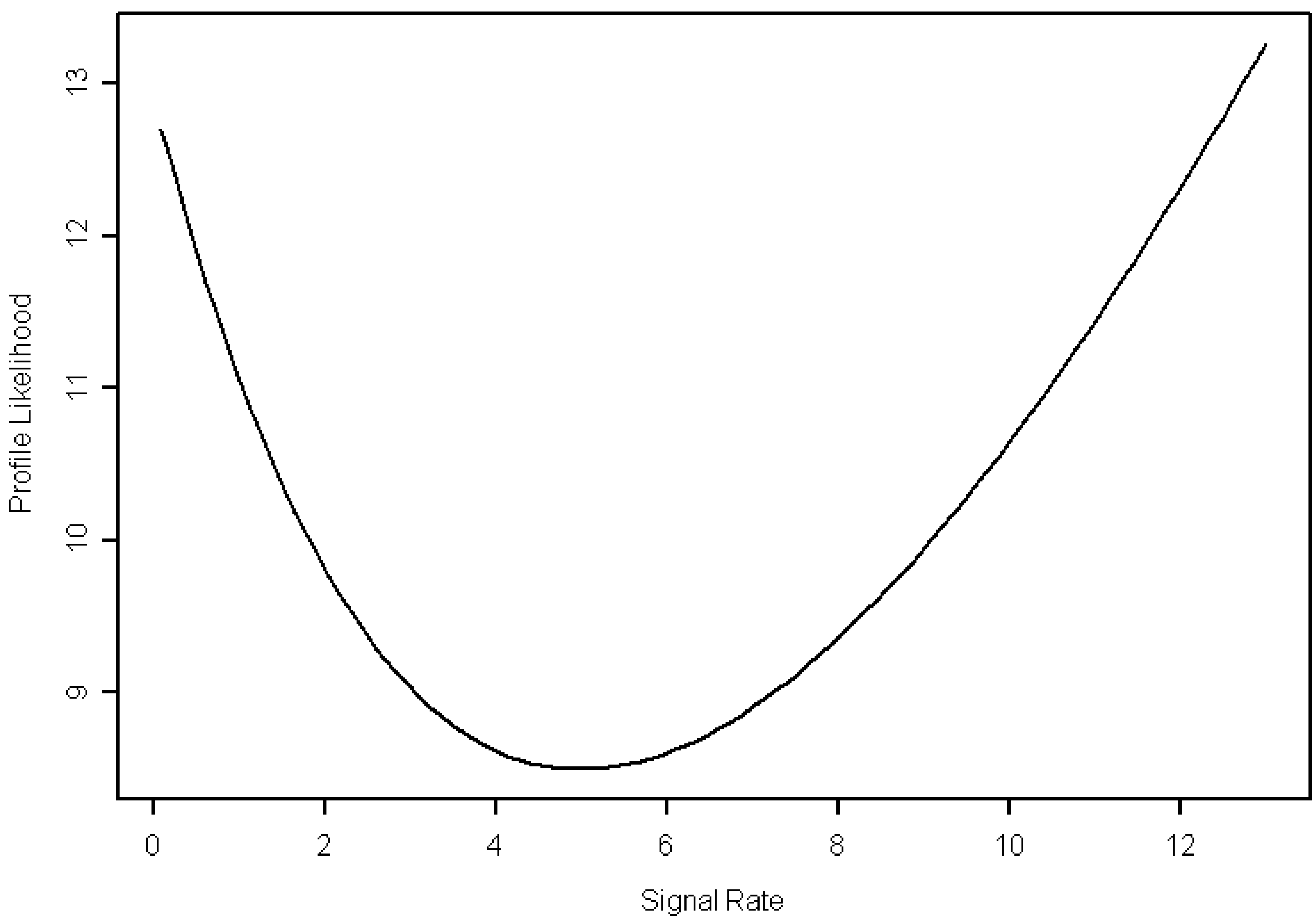
- Case: x=8
- y=15
- tau=5.0
- e=100%

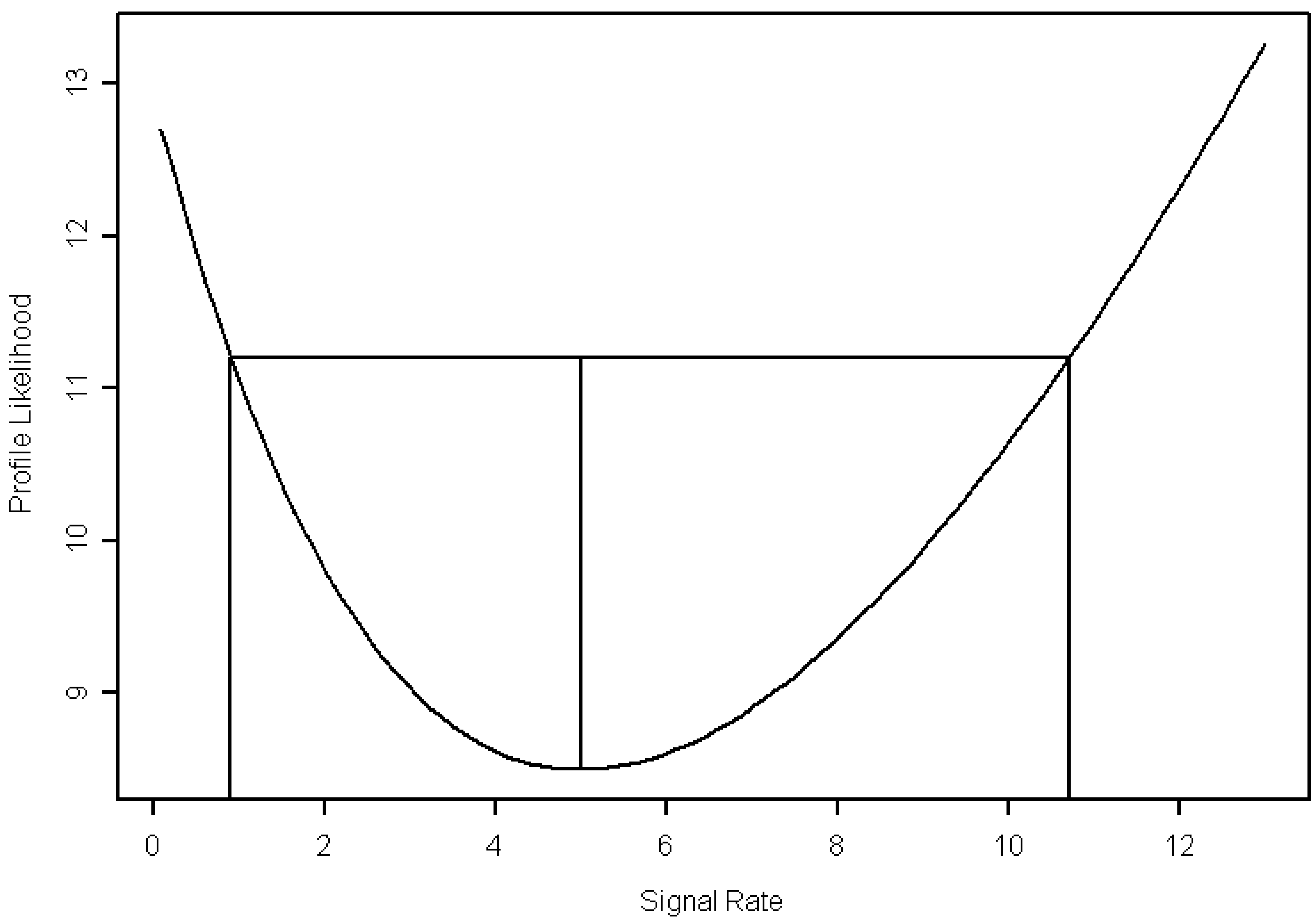
(known)
- mu fixed at 2

→ bhat = 3.33




Z
25
20
15
10
5
0
5
4.5
4
3.5
3
Signal Rate
2
4
6
8
10
12


Profile Likelihood
13
12
11
10
9
0
2
4
6
8
10
12
Signal Rate



Profile Likelihood
13
12
11
10
9
0
2
4
6
8
10
12
Signal Rate

Sometimes this can be done analytically, sometimes (like here) it has to be done numerically.

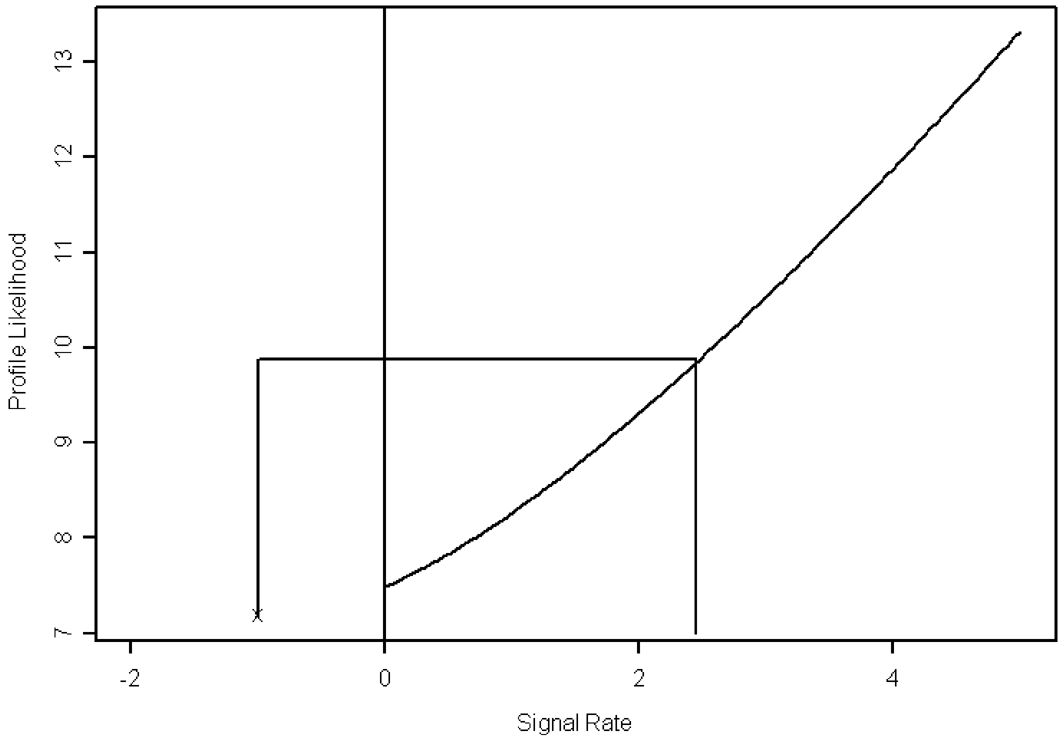
Result: given the data (x,y,z,τ,m) the profile likelihood is a function of μ alone

$\rightarrow$ no more nuisance parameters

One Problem: $x<y/\tau$

- Then mle of $\mu < 0$
- Example: same as before, but x=2, so

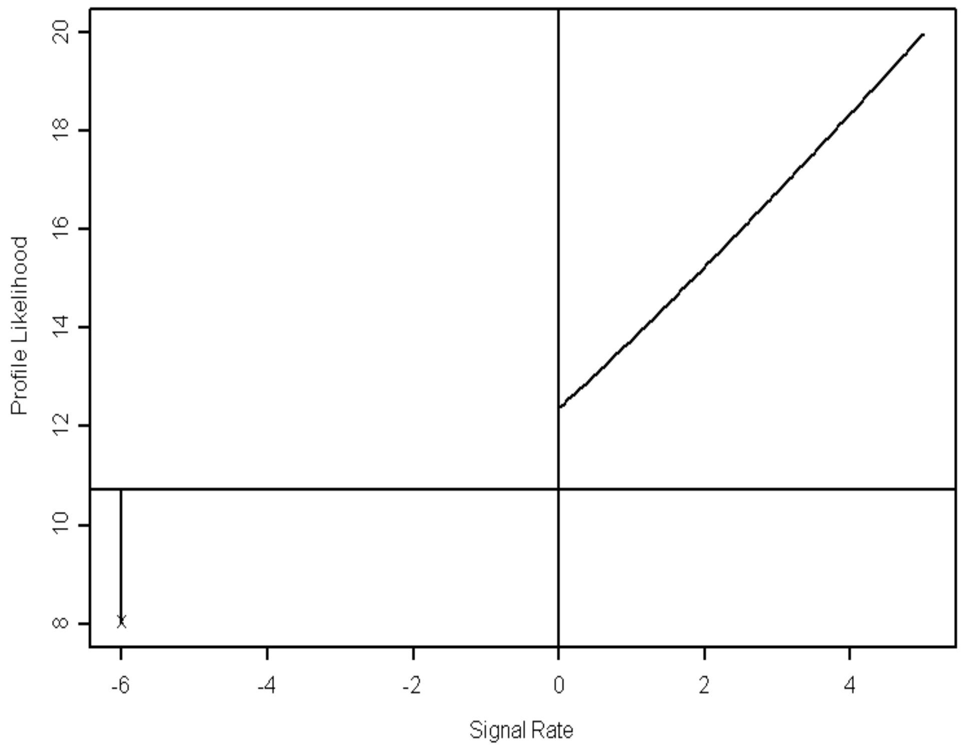
  $x-y/\tau = -1.0$
- 90% upper limit is 2.45



Even worse:

- Same as last, but y=35.0
- So we expect 7 events just from background, but we only see 2

Note: even if μ=0 this happens only about 5% of the time.



Two ways to handle this:

- keep y, z, τ, m fixed, find smallest x for which upper limit is greater than 0

  → intuitive meaning of "upper limit"

  "unbounded likelihood method"

- use constrained likelihood, i.e. require mle ≥ 0 always

  → uses physical limits on parameters

  "bounded likelihood method"

Method can deal with other situations:

- Background and/or Efficiency are known without error

- Background is Gaussian instead of Poisson:
  $y \sim N(b,\sigma_b)$

- Efficiency is Gaussian instead of Binomial:
  $z \sim N(e,\sigma_e)$

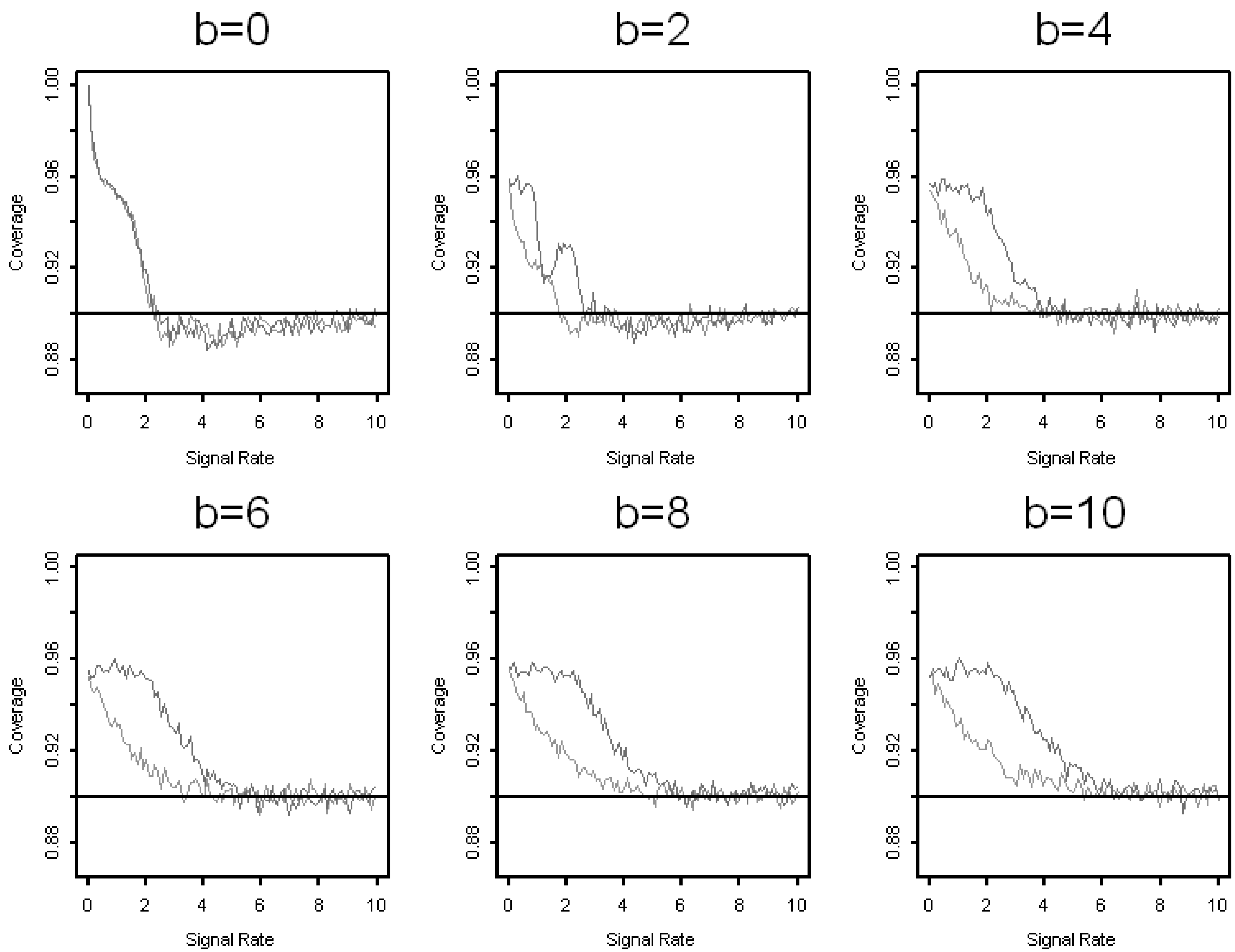
→ Allows incorporation of systematic errors

So, does it work?

- Confidence Intervals work if they have coverage:
- Fix μ, b, e, σ_b, σ_e and α
- Generate $y_1,..,y_n \sim N(b, \sigma_b)$
- Generate $z_1,..,z_n \sim N(e, \sigma_e)$
- Generate $x_1,..,x_n \sim Pois(e\mu + b)$
- Find $(1-\alpha)100\%$ CI's (L_i,U_i) for i=1,..,n
- Find percentage p with $L_i \leq \mu \leq U_i$
- If $p \geq (1-\alpha)100\%$, we have correct coverage
- Repeat for many values of μ, b, e, σ_b, σ_e and α

Example

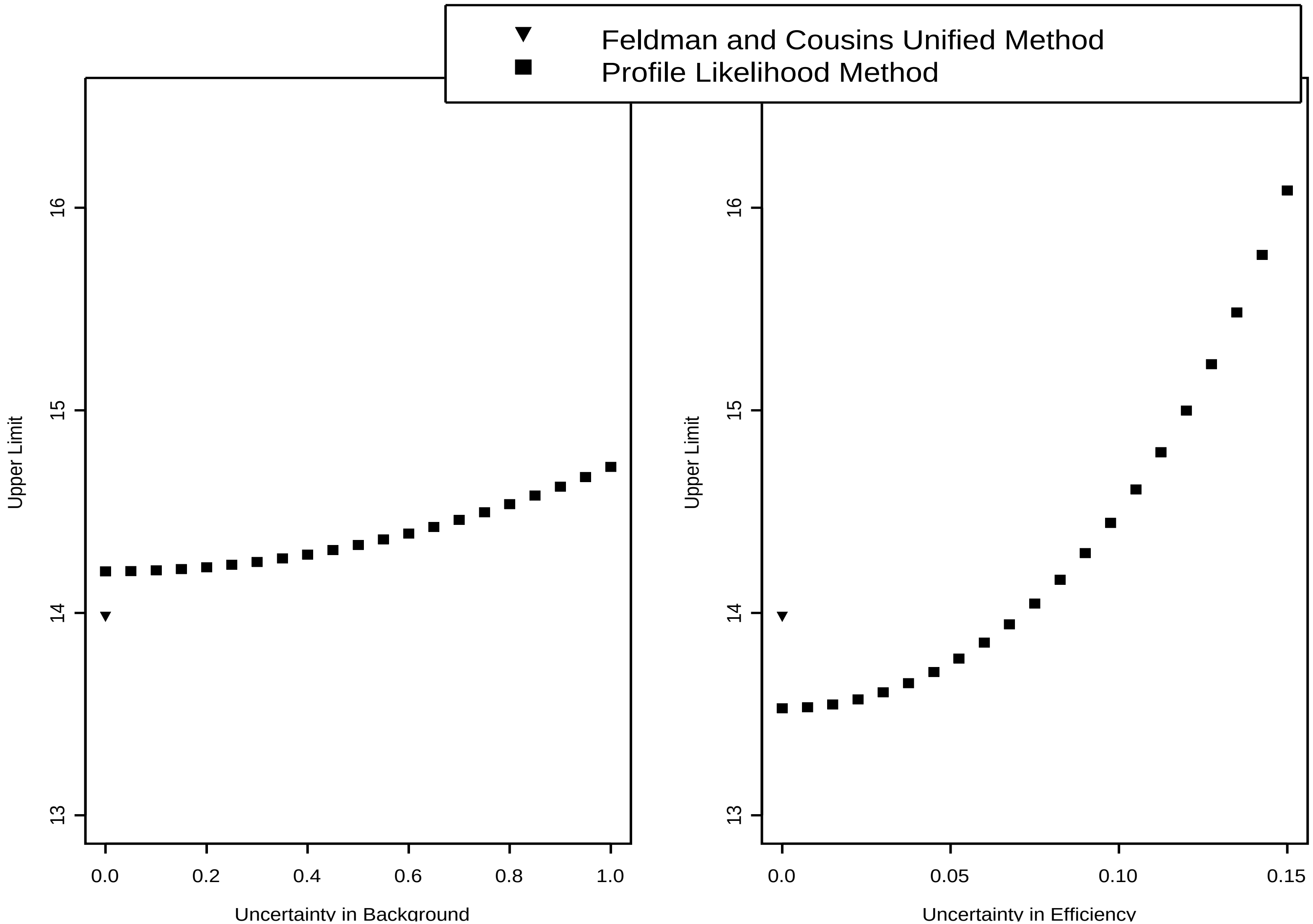
- Background – Gaussian with error 0.5
- Efficiency – Gaussian with mean 0.85 and error 0.075
- Signal rate varies from 0 to 10 in steps of 0.1
- Background rate varies from 0 to 10 in steps of 2
- Nominal coverage rate 90%
- Orange – unbounded likelihood
- Blue - bounded likelihood


b=0
b=2
b=4
b=6
b=8
b=10
Coverage
Signal Rate
1.00
0.96
0.92
0.88
0
2
4
6
8
10

Features of our Method:

- Always yields positive upper limit
- Smooth transition from upper limits to two-sided intervals
- Now available as part of ROOT: TRolke
- Limits are consistent as errors on nuisance parameters become small:

TRolke Intervals



Isn’t it a marvelous new method ?

- See F. James, MINUIT Reference Manual, CERN Library Lo

”The MINO ange in the value of the mount UP, where F’ is t

NO ! It is a marvelous old method … nobody knew how marvelous though

Confidence Interval

Profile Likelihood (in X^2 approximation)

$\Delta X^2 = 2.71$ (90%), $\Delta X^2 = 1.07$ (70 %)

Summary

- Profile Likelihood is a general technique for dealing with nuisance parameters
- It is familiar to physicists as part of MINUIT
- For the problem of setting limits for rare decays it yields a method with good coverage and some nice properties
- It is available as part of ROOT
- The End